

RANDOM FIELD ESTIMATION APPROACH TO ROBOT DYNAMICS

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ABSTRACT

Random field models provide an alternative to the deterministic models of classical mechanics used to describe multibody robot arm dynamics. These alternative models can be used to establish a relationship between the methodologies of estimation theory and robot dynamics. A new class of algorithms for such fundamental robotics problems as inverse dynamics, inverse kinematics, forward dynamics, etc. can be developed that use computations typical in estimation theory. The central result is an equivalence between inertia and covariance. This allows much of what is known about covariance factorization and inversion to be used for inertia matrix inversion. In particular, it is known that the difference equations of Kalman filtering and smoothing factor and invert recursively the covariance of the output of a linear state-space system driven by a white-noise process. Here, it is shown that similar recursive techniques factor and invert the inertia matrix of a multibody robot system. The random field models are based on the assumption that all of the inertial (D'Alembert) forces in the system are represented by a spatially distributed white-noise model. They are easier to describe than the models based on classical mechanics, which typically require extensive derivation and manipulation of equations of motion for complex mechanical systems. In contrast, with the spatially random models, more primitive (i.e., simpler and less dependent on mathematical derivations) locally specified computations result in a global collective system behavior (as represented by the inertia matrix) equivalent to that obtained with the deterministic models. The primary goal in investigating robot dynamics from the point of view of random field estimation is to provide a concise analytical foundation for solving robot control and motion planning problems.

1 Introduction and Summary

This paper examines from the point of view of random field estimation the methods developed recently by the author [1]–[9] to solve the problem of forward dynamics for nonlinear joint-connected robot arms and multibody systems. The forward dynamics problem is to find the joint-angle accelerations from the applied joint moments. Solution of this problem is of interest in such application areas as robotics simulation and control design. The problem is solved by the recursive filtering and smoothing techniques of state estimation theory [10, 11] for linear, discrete-time, state-space dynamical systems. The filtering stage takes the applied joint moments as inputs (measurements in the Kalman filter) to produce a sequence of spatial (rotational and translational) constraint forces acting at the joints of the system. The smoothing stage takes the innovations process resulting from the filter as an input and produces a set of spatial accelerations and a corresponding set of joint-angle accelerations.

One of the reasons for using filtering and smoothing techniques is to provide what is believed to be a better means to formulate, analyze and understand multibody robot dynamics. It is known that the relationship between joint-angle accelerations at any given time and the joint moments applied at the same time is an affine relationship. One of the initial steps of the paper is to develop a spatially recursive state space model to characterize this relationship. Development of the state space model makes it possible to apply many of the ideas and concepts (transition matrix, prediction, filtering, smoothing, etc.) from the state estimation theory for linear systems. These concepts have proven themselves to be ideally suited to investigate discrete-time systems. They are also very useful in solving multibody robot dynamics problems. The filter and smoother are very easy to understand, and extensive analytical and computational experience exists with this architecture in other application areas. Standardized software also is available that can be used to set up readily the required computations.

The table in Fig. 1 shows in perspective the context in which the random field estimation approach fits within three other approaches to multibody robot dynamics. Outlined on the extreme left of the figure is the approach of Newton-Euler, in which Newton's laws for a single particle are used as a starting point. These laws are combined with the geomet-

rical constraint that there is no relative motion between two points on the same rigid body. Constraint forces between bodies are then eliminated in order to obtain a set of equations of motion of the form

$$\mathcal{M}(\theta)\ddot{\theta} = T$$

in which θ are the joint angles; $\ddot{\theta}$ are the corresponding joint-angle accelerations; T are the applied moments; and \mathcal{M} is the composite multibody system inertia matrix. For simplicity, without loss of generality [1], the forces and accelerations due to nonlinear velocity dependent terms are not included in this equation. An alternative approach, shown on the second column of the diagram, is based on Lagrangian mechanics, which consists of finding an extremum for the Lagrangian of the system. The first variation of the Lagrangian leads to the Euler equation as a necessary condition for extremality. Application of this equation to the multibody problem leads to the equations of motion. These equations are exactly the ones obtained by the Newton-Euler approach. The Lagrange and Newton-Euler approaches are therefore equivalent. However, the Lagrangian approach involves a higher-level formulation, in the sense that the statement of the approach is simpler (with simpler assumptions). It also does not require that constraint forces and moments be eliminated by manipulation of equations. This elimination takes place automatically. This is in contrast with the Newton-Euler approach, in which the constraint forces and moments between bodies must be eliminated from the equations of motion. Both the Newton-Euler and Lagrange approaches lead to a set of equations of motion in which the composite system inertia matrix \mathcal{M} is present. However, there is nothing inherent in the multibody dynamics problem that makes the presence of such a matrix inevitable. The reason this matrix is at times thought to be indispensable in multibody dynamics problems is that the most popular approaches (Newton-Euler and Lagrange) for derivation of equations of motion involve this matrix.

The approach of this paper, outlined on the third column of the diagram, leads to a set of equations of motion in which the composite inertia matrix is never assembled. Instead, the joint-angle accelerations are computed recursively in terms of the applied joint moments. There is no intermediate step in which the inertia matrix is computed and then inverted. It can be shown [1] that this recursive process is equivalent to a set of equations of motion of

the form

$$\ddot{\theta} = \mathcal{M}^{-1}(\theta)T$$

in which the right hand side is evaluated directly, without resorting to numerical assembly and inversion of the matrix. This is distinctly different from both the Newton-Euler and Lagrange approach, in which the inertia matrix appears on the left side of the equations of motion.

The random field estimation approach is at a higher level in the same sense that Lagrangian mechanics is at a higher level than Newtonian mechanics. The estimation problem involves optimization in the same sense that the Lagrangian approach involves optimization. The estimation approach starts out with a spatially random model and proceeds to the development of a Fredholm matrix equation as a necessary and sufficient condition for optimality. Solution of this Fredholm equation requires the inverse of a covariance matrix, which is equivalent to the composite multibody robot system inertia matrix. Recursive factorization and inversion of this matrix is achieved by the filtering and smoothing algorithms of this paper.

Closely related to the random-field estimation approach is a two-point boundary-value problem and its solution, outlined in the fourth column of Fig. 1. The two-point boundary-value problem is very similar to those which result as necessary and sufficient conditions for optimality in quadratic optimization problems [1, 11]. The approach involving a two-point boundary-value problem, studied in detail in [1], bears the same relationship to the random field estimation method of this paper that Newton-Euler classical mechanics bears to Lagrangian mechanics.

The main motivation for the author's interest in the random-field estimation approach to robot modeling is that it leads to a high-level mathematical framework for describing the complex mechanics of a multibody robot system. This framework, referred to as a spatial operator algebra, is used to some extent in this paper and is more completely developed in [8]. One of its key features is that it reduces by at least an order of magnitude the number of symbols and operations that the user has to see in order to solve robot modeling problems.

A concrete example of this ability to solve complex problems with simple spatial operator notation is the closed-form evaluation of the manipulator inertia matrix contained in Sec. 11. Another feature of the spatial operator algebra is that high-level operator equations can be converted by inspection into spatially recursive algorithms for performing the required computations. Thus, every mathematical expression involving operators has a corresponding equivalent spatially recursive mechanization where the number of arithmetical operations increases only linearly with the number of degrees of freedom.

The ultimate objective is to solve problems in automated robot motion planning and control. The solution to these problems are typically based on robot models. Random field estimation provide a mathematical foundation to concisely state and solve these problems. Applications to decoupled robot control are discussed in [27]. Statistical mechanics models that systematically account for interactions with objects in the environment are discussed in [28]. Application of these statistical models to the problem of motion planning are currently under investigation.

2 Configuration and Problem Statement

Three typical systems formed by rigid bodies connected by joints are illustrated in Fig. 2. These systems are models for a wide class of robotic manipulator and end-effector systems. The serial chain of Fig. 2 (a) can be used to represent a single manipulator. The topological tree of Fig. 2 (b) can represent several arms mounted on a mobile robot. The topological graph of Fig. 2 (c) can represent several arms or fingers moving a commonly grasped object. The serial chain system is used in this paper to develop the random field estimation approach. Topological trees and closed-chain systems are analyzed in [3, 5, 6] by the same methods.

The serial chain system has N bodies numbered $1, \dots, N$. It is connected together by N joints also numbered $1, \dots, N$. Joint N is the last in the sequence, and it connects body N to an immobile base. As a different option, joint N could also be a fictitious joint that models the motion of the base body with respect to an inertial frame of reference. A given rigid body k in the collection is characterized by a mass $m(k)$, a vector $p(k)$ from the inner

joint k to the body k mass center, an inertia matrix $\mathcal{J}(k)$ about the inner joint, a vector $l(k, k-1)$ from the inner to the outer joint, and a unit vector $h(k)$ along the axis of rotation.

The forward dynamics problem is to find the joint angle accelerations $\ddot{\theta}(k)$, given the applied joint moments $T(k)$.

3 State Space Model

It is known that, at a given instant in time, the relationship between applied moments and the resulting accelerations is affine [1]. Recall that an affine mapping relating the quantities x and y is of the form $y = \ell x + c$, in which ℓ is a linear operation and c is a constant. Therefore an affine mapping is very similar to a linear mapping and differs from it only because of the presence of the constant term c . One of the key ideas of this paper is to use a state space model to characterize this affine relationship. This approach allows the application of highly developed techniques from linear system theory to solve multibody robot dynamics problems. It is interesting to note that the linear system theory applies, although the multibody robot system is highly nonlinear.

The first important step leading to the filtering and smoothing methods is to develop a spatially recursive state space model for the dynamics of each body. Consider a typical body k as shown in Fig. 3.

The states in the model are the 6-dimensional spatial forces $f(k) = [N^*(k), F^*(k)]^*$ consisting of the three moments $N(k)$ and the three linear forces $F(k)$. The symbol $f(k)$ denotes the force acting on body k at joint k due to the adjoining body $k+1$. Similarly, the symbol $f(k-1)$ denotes the spatial force acting on body $k-1$ at joint $k-1$ and due to the adjoining body k . Spatial forces are propagated from the outer joint $k-1$ to the inner joint k by means of a 6×6 transition matrix

$$\phi(k, i) = \begin{pmatrix} I & \tilde{l}(k, i) \\ 0 & I \end{pmatrix}$$

defined in terms of the vector $l(k, i)$ from joint k to joint i . The cross-product operation

with the vector $l(k, i)$ is denoted by the 3×3 matrix $\tilde{l}(k, i)$. The matrix $\phi(k, i)$ satisfies the properties

$$\phi(k, i) = \phi(k, m)\phi(m, i); \quad \phi^{-1}(k, i) = \phi(i, k); \quad \phi(k, k) = I$$

usually associated with a transition matrix for a state space linear system.

Costates in the model are the spatial accelerations $\alpha(k) = [\dot{\omega}^*(k), \dot{v}^*(k)]^*$, in which $\dot{\omega}(k)$ and $\dot{v}(k)$ are the inertial time derivatives of the angular and linear velocities $\omega(k)$ and $v(k)$. The spatial acceleration at joint k is defined on the outboard (toward the tip of the system) side of joint k .

The inertia properties of the typical body k in Fig. 3 are embedded in the following spatial inertia matrix [1]

$$M(k) = \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I \end{pmatrix}$$

This is a 6×6 matrix whose diagonal elements depend on the rotational inertia $\mathcal{J}(k)$ and on the mass $m(k)$, and whose off-diagonal elements depend on the mass and on the vector $p(k)$ from joint k to the mass center of body k . The spatial inertia matrix very succinctly accounts for the rotational inertia and mass properties of body k about joint k .

4 Two-Point Boundary-Value Problem

The equations of motion for a robot arm are defined by the following two-point boundary-value problem:

$$\left\{ \begin{array}{l} f(0) = 0 \\ \text{for } k = 1 \dots N \\ \quad f(k) = \phi(k, k-1)f(k-1) + M(k)\alpha(k) + b(k) \\ \quad T(k) = H(k)f(k) \\ \text{end loop} \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha(N+1) = 0 \\ \text{for } k = N \dots 1 \\ \alpha(k) = \phi^*(k+1, k)\alpha(k+1) + H^*(k)\ddot{\theta}(k) \\ \text{end loop} \end{array} \right.$$

This problem is very similar to those investigated extensively in the areas of quadratic optimal estimation and control [11]. The problem is defined in terms of states and costates. Boundary conditions are satisfied at two distinct points: the base of the system, where the accelerations vanish; and the terminal body, where the forces vanish. These free-fixed boundary conditions correspond to an immobile base body. Other boundary conditions can be handled easily within the same general framework [3, 5, 6, 30]. For example, free-free boundary conditions (in which both the base body and the terminal bodies are free to accelerate) can be stated by imposing suitable constraints on the states and costates. Such free-free boundary conditions are suitable for applications in which the base of the system is unattached.

There is a one-to-one map between the boundary-value problem just described and those of estimation theory [1]. This map is illustrated in Fig. 4. The spatial forces in the dynamics problem are the states of the system. The spatial accelerations are the costates. The applied joint moments correspond to the measurements or the outputs of the system. The equivalence between measurements and applied moments makes sense because: measurements are those quantities that are assumed to be known in the estimation problem; whereas applied joint moments are assumed to be known in the dynamics problem. The transition matrix for a discrete-time state space system is typically defined in terms of the time interval between samples. The transition matrix for the dynamics problem is defined in terms of the spatial interval, the 3-dimensional vector from the inner to the outer joint in a given body. This matrix can be used to propagate inwardly the forces within a body and its transpose can be used to propagate outwardly the velocities and accelerations. The use of the term intrabody Jacobian for this matrix is an extension of terminology used in robotics. The process error covariance in estimation problems corresponds to the spatial inertia. Known deterministic inputs correspond to the bias spatial force due to nonlinear gyroscopic and velocity dependent effects. The state-to-output map projects the spatial force

at any given joint into the active joint moment applied at the same joint.

It is known [11] that boundary-value problems of the type just described can be solved by means of Kalman filtering and smoothing. An easy way to show this is to use what is referred to in [11] as the sweep method. This method begins with the assumption that the states $f(k)$ and the costates $\alpha(k)$ are related by

$$f(k) = z(k) + P(k)\alpha(k)$$

In this relationship, $z(k)$ plays the role of the filtered state estimate in the Kalman filter, and $P(k)$ plays the role of the corresponding filtered state estimation error covariance. The filtered state estimate is generated by a spatial Kalman filter. The estimation error covariance is generated by the Riccati equation. The spatial accelerations $\alpha(k)$ are generated by a Bryson-Frazier smoother. These results can be obtained by substituting the above relationship between states and costates into the two-point boundary-value problem above. Alternatively, they can be developed using the recursive mass matrix factorization and inversion results of the following section.

The equations of motion are now written in a more compact form. We define M as the $6N \times 6N$ block diagonal matrix defined as $M = \text{diag}[M(1), \dots, M(N)]$ whose typical 6×6 diagonal block $M(k)$ is the spatial inertia of body k about joint k . The matrix ϕ is a causal (lower triangular) matrix defined as

$$\mathcal{E}_\phi \triangleq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \phi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \phi(n,n-1) & 0 \end{pmatrix}, \quad \phi = (I - \mathcal{E}_\phi)^{-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \phi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(N,1) & \phi(N,2) & \dots & I \end{pmatrix}$$

The composite state-to-output map H in Eq. (5.4) is defined as $H = \text{diag}[H(1), \dots, H(N)]$. The appropriately dimensioned identity is denoted by I throughout this paper. Then,

$$V = \phi^* H^* \dot{\theta}$$

$$\alpha = \phi^* H^* \ddot{\theta}$$

$$f = \phi M \alpha = \phi M \phi^* H^* \ddot{\theta}$$

$$T = H f = \mathcal{M} \ddot{\theta}$$

where,

$$\mathcal{M} = H \phi M \phi^* H^*$$

Here, \mathcal{M} is the robot inertia (mass) matrix. The composite multibody robot inertia matrix can be used to obtain the kinetic energy of the robot useful in the methods of classical mechanics. The total kinetic energy in the system is given by

$$K.E. = \frac{1}{2} \sum_{k=1}^N V^*(k) M(k) V(k)$$

in which $V(k)$ is the spatial velocity on the negative side (outboard, toward the tip) of joint k , and $M(k)$ is the spatial inertia of body k about joint k . More succinctly, the kinetic energy can be expressed as

$$K.E. = \frac{1}{2} V^* M V$$

in which $V = [V(1), \dots, V(N)]$ is the composite vector of spatial velocities. Recall, however, that the spatial velocities are given by

$$V = \phi^* H^* \dot{\theta}$$

in terms of the joint-angle velocities $\dot{\theta}$. This leads to

$$K.E. = \frac{1}{2} \dot{\theta}^* \mathcal{M} \dot{\theta}$$

5 Spatially Random Model

The point of departure for the random field estimation approach to robot dynamics is the following state-space model for a typical body k in the system

$$X(k) = \phi(k, k-1)X(k-1) + W(k) \quad (5.1)$$

$$T(k) = H(k)X(k) \quad (5.2)$$

in which the moments $T(k)$ are due to external sources acting at the joints. The state $X(k)$ of the system is a 6-dimensional vector of spatial forces, formed by three moments and three linear forces. The above is a linear model that reflects a balance of the forces that are acting on body k . The inertial forces are represented by a spatially distributed white-noise process $W(k)$ whose mean and covariance are

$$E[W(k)] = 0; \quad E[W(k)W^*(k)] = M(k) \quad (5.3)$$

While the mean value of the inertial force is assumed to be zero here, in general it may be set equal to the bias force which accounts for nonlinear gyroscopic and velocity dependent terms, as well as effects due to external forces acting on the body [1]. The covariance of the inertial force is set equal to the spatial inertia matrix $M(k)$. The output, or measurement, Eq. (5.2) completes a description of the spatially random model. In this model, the active joint moment $T(k)$ plays the role of the measurement in a linear state space system. Since the joint moments are known exactly, the corresponding measurement equation Eq. (5.2) is free of measurement noise.

This model can be used for one-step prediction of the spatial force at the inner joint of a body, given the spatial force at the outer joint. The model has a built-in error represented by the term $W(k)$ in this equation. This error is due to the fact that the body is in general moving and undergoing acceleration. This acceleration causes a D'Alembert force $W(k)$.

The model in Eq. (5.1)-Eq. (5.3) above takes the point of view of a local observer whose perception is by definition confined to the location of joint k itself. This observer

is assumed to know nothing about the applied moments acting at the joints or about the accelerations acting on any of the adjoining bodies. The observer assumes that its own acceleration at joint k is uncorrelated with the acceleration at the remaining joints. It should be pointed out that the white-noise model is only an "a priori" model. There is uncertainty in the inertial forces only before the estimation process is conducted. After estimation takes place, this uncertainty no longer exists.

The above model can be cast in the more compact notation

$$X = \phi W; \quad T = HX \quad (5.4)$$

in terms of the composite vectors $W = [W(1), \dots, W(N)]$, $X = [X(1), \dots, X(N)]$ and $T = [T(1), \dots, T(N)]$. The composite process error vector W has a mean and covariance given by

$$E(W) = 0; \quad E[WW^*] = M$$

The model in Eq. (5.4) is based on the assumption that the D'Alembert force due to the acceleration of body k is an equivalent lumped quantity that is concentrated at the joint k . In reality, there is an inertial force at every location x of the spatial domain Ω of definition of the system. The D'Alembert force within a given body is therefore spatially distributed over the body. To take into account this spatial distribution, it is possible to recast Eq. (5.4) as the following integral operator model:

$$T(k) = \int_{\Omega} H(k)\phi(k, x)Bw(x)dx; \quad T = H\phi Bw$$

in which Ω is the spatial domain of definition of the system; $w(x)$ is the D'Alembert force at a given spatial location x ; and B is the 3×6 matrix $B = [0, I]^*$. The D'Alembert force is a white-noise random field characterized by its mean value and covariance

$$E[w(x)] = 0; \quad E[w(x)w^*(y)] = \rho(x)\delta(x - y)$$

where $\rho(x)$ is the mass density at location x , and $\delta(x - y)$ is the impulsive delta function.

In operator notation, this becomes

$$E(w) = 0; \quad E[ww^*] = \rho I$$

This integral operator model, in which the inertial forces are modeled as a random field, is more fundamental than the equivalent model in Eq. (5.3) in which the inertial forces have been lumped at the joint k . The integral operator model views the inertial force at an arbitrary point x of the system as an input. The output of the model is the active joint moment at a given joint k . The input at x and the output at k are related by the kernel $H(k)\phi(k, x)B$ of the integral operator $H\phi B$. The active joint moment $T(k)$ is the superposition of the D'Alembert forces at all of the points x in the system. This relationship is linear, even though the original system is highly nonlinear.

6 Equivalence Between Covariance and Inertia

One of the central results of the paper, is to establish an equivalence between the composite multibody robot system inertia matrix and the covariance of the output of the linear system model just described.

Lemma 1: The state covariance matrix $S = E[XX^*] = \phi M \phi^*$ can be expressed as

$$S = R + \tilde{\phi}R + R\tilde{\phi}^* \tag{6.1}$$

where

$$\tilde{\phi} \triangleq \phi - I$$

The $6N \times 6N$ block-diagonal matrix $R = \text{diag}[R(1), \dots, R(N)]$ has blocks $R(k)$ which are

generated recursively by

$$\left\{ \begin{array}{l} R(0) = 0 \\ \text{for } k = 1 \dots N \\ R(k) = \phi(k, k-1)R(k-1)\phi^*(k, k-1) + M(k) \\ \text{end loop} \end{array} \right. \quad (6.2)$$

Proof: Observe that Eq. (6.2) can be rewritten in the form

$$M = R - \mathcal{E}_\phi R \mathcal{E}_\phi^*$$

Pre and post multiplying this by ϕ and ϕ^* leads to

$$S = \phi R \phi^* - \phi \mathcal{E}_\phi R \mathcal{E}_\phi^* \phi^* = (\tilde{\phi} + I)R(\tilde{\phi} + I)^* - \tilde{\phi} R \tilde{\phi}^*$$

The result follows from expanding this. ■

This implies im that the typical element kernel $R(k, j)$ of S is given by

Typical Element of State Covariance Matrix		
	$j \leq k$	$j \geq k$
$R(k, j)$	$\phi(k, j)R(j)$	$R(k)\phi^*(j, k)$

The mean value and the covariance of the output T of the linear system in Eq. (5.4) is given by

$$E(T) = 0; \quad E[TT^*] = \mathcal{M}$$

which follows because the system in Eq. (5.4) is a linear system and because $E(W) = 0$ and $E[WW^*] = M$. This implies the central result of this section.

Lemma 2: The covariance of the output of the linear system model in Eq. (5.4) equals

the composite multibody robot system inertia, *i.e.*,

$$E[TT^*] = \mathcal{M} \tag{6.3}$$

Hence, the inertia matrix can be factored and inverted using techniques that factor and invert the covariance matrix.

This result has profound implications. It establishes the equivalence between inertia, one of the key concepts in mechanics, and covariance, one of the key concepts in probability and statistics. There is a fundamental physical reason for this which is investigated in more detail in [1]. An immediate and practical implication of the result is: the inertia matrix can be inverted by filtering and smoothing methods that have been developed to invert covariance matrices [12]–[21]. Much of this background has been developed for problems in communication theory, information theory and signal processing. The references provide a sample of these results. One of the central ideas which emerges from these references is that the covariance (or, equivalently, the multibody body inertia matrix) can be factored as $\mathcal{M} = (I + \mathcal{K})D(I + \mathcal{K}^*)$ in which \mathcal{K} is a lower-triangular “causal” matrix and \mathcal{K}^* is its anticausal adjoint. The matrix D is diagonal. This is strongly reminiscent of the Gohberg-Krein factorization [13]. Both the matrix \mathcal{K} and D can be generated by means of a Kalman filter. This result underlies much of today’s research [19] on linear filtering and prediction theory.

7 Conditional Mean Estimation

The estimation problem to be solved here is to estimate the process error vector W and the state X , given all the measurements T . This corresponds to the dynamics problem of finding the inertial forces (due to accelerations) and the spatial forces, given the joint moments. The optimal estimates are obtained by means of the conditional expectations $E(W/T)$ and $E(X/T)$. It is relatively simple to compute these two conditional expectations.

By methods outlined in [12], it can be established that

$$E(X/T) = \mathcal{G}T \quad (7.1)$$

in which \mathcal{G} is the “Kalman” gain

$$\mathcal{G} \triangleq SH^*M^{-1} \quad (7.2)$$

This is the estimate of the spatial forces given the applied joint moments. The effect of measurements is accounted for in the term involving the Kalman gain in Eq. (7.2). The Kalman gain determines the weight of the observation to arrive at the state estimate $E(X/T)$. Observe that the $N \times N$ matrix \mathcal{M} that needs to be inverted to compute the Kalman gain is the composite multibody robot system inertia matrix defined in Sec. 4.

To examine the analogy more closely, recall that the joint-angle accelerations are

$$\ddot{\theta} = \mathcal{M}^{-1}T \quad (7.3)$$

The joint-angle accelerations $\ddot{\theta}$ and the spatial accelerations $\alpha = [\alpha(1), \dots, \alpha(N)]$ at the N joints are related by [1]

$$\alpha = \phi^* H^* \ddot{\theta} \quad (7.4)$$

Thus, the estimate $E[X/T]$ is given by

$$E[X/T] = \phi M \alpha = f$$

That it is precisely the spatial force f at the joints, and from now on we adopt the terminology f when referring to the estimate $E[X/T]$.

It is of interest to distinguish between the “fictitious” spatial force X emerging from the spatially random model and the corresponding conditional mean estimate f . These two quantities are very closely related, but they are not identical. They have the same physical units. Furthermore, $X(k)$ and $f(k)$ are defined at the same joint k . They differ because $X(k)$ is a “fictitious” random quantity in the sense that it emerges from the spatially random

model, whereas $f(k)$ is a deterministic estimate based on the information about the joint moments T . Solutions to dynamics problems seek to compute the estimate f . This estimate corresponds to the spatial forces acting in the physical mechanism. The random state vector $X(k)$ is a quantity that is useful in setting up the spatially random model but which does not correspond to the forces present in the physical mechanism.

To compute the covariance of the estimation error observe first that

$$e_s \triangleq X - f = (I - \mathcal{G}H)\phi W$$

is the estimation error. Its corresponding covariance is

$$\mathcal{P} = E[e_s e_s^*] = (I - \mathcal{G}H)S(I - \mathcal{G}H)^* \quad (7.5)$$

Alternatively, this becomes

$$\mathcal{P} = (I - \mathcal{G}H)S = S(I - \mathcal{G}H)^* = S - SH^* \mathcal{M}^{-1}HS$$

Note that $H\mathcal{P} = 0$, $\mathcal{P}H^* = 0$, $H\mathcal{P}H^* = 0$ which imply that the covariance of the estimation error at the joints vanishes. This reflects the lack of measurement noise in the measurement Eq. (5.2).

The conditional-mean estimate for the inertial forces is given by

$$E(W/T) \triangleq \hat{W} = M\phi^* H^* \mathcal{M}^{-1}T$$

Continuing the dynamics analogy, the estimate for the inertial forces becomes

$$\hat{W} = M\alpha$$

The covariance of the inertial force estimation error is obtained by arguments very similar to those used to arrive at Eq. (7.5). Observe first that $W - \hat{W} = [I - M\phi^* H^* \mathcal{M}^{-1}H\phi]W$ is

the estimation error. The error covariance Q is

$$Q = M - M\phi^*H^*\mathcal{M}^{-1}H\phi M$$

The foregoing are “batch” solutions to the estimation problem, in the sense that all of the measurements are processed simultaneously. This implies that the composite system inertia matrix must be inverted in a batch mode. Alternatives are the sequential filtering and smoothing solutions outlined in the following two sections.

8 Sequential Estimation

Sequential solutions [1] process the measurements (the applied moments) one at a time. In doing this, numerical inversion of the $N \times N$ system inertia matrix is not required. Instead, the inertia matrix is factored as

$$\mathcal{M} = (I + \mathcal{K})D(I + \mathcal{K}^*) \quad (8.1)$$

in which D is an $N \times N$ diagonal matrix, and \mathcal{K} is a lower-triangular matrix. The matrices K and D in this factorization are generated using a suitably defined Kalman filter. This factors the inertia into the product of a causal factor $(I + \mathcal{K})$, a diagonal matrix D , and the anti-causal adjoint factor $(I + \mathcal{K}^*)$. Once this factorization of the system inertia matrix is achieved, the corresponding inverse can be computed easily. The central result is that

$$(I + \mathcal{K})^{-1} = I - \mathcal{L}$$

where \mathcal{L} is a lower-triangular causal matrix generated by the same Kalman filter that generates \mathcal{K} . This implies that the inertia matrix inverse can be expressed as

$$\mathcal{M}^{-1} = (I - \mathcal{L}^*)D^{-1}(I - \mathcal{L}) \quad (8.2)$$

To arrive at the factorizations in Eq. (8.1) and Eq. (8.2) requires some preliminaries which are outlined below.

The following Kalman filtering equations define $P(k)$ which satisfies the discrete Riccati equation, as well as the innovations covariance $D(k)$, the Kalman gains $G(k)$ and $K(k+1, k)$, the projection operators $\tau(k)$ and $\bar{\tau}(k)$, and the transition matrix $\psi(k+1, k)$:

$$\left\{ \begin{array}{l} P(0) = 0 \\ \text{for } k = 1 \dots N \\ \quad P(k) = \psi(k, k-1)P(k-1)\psi^*(k, k-1) + M(k) \\ \quad D(k) = H(k)P(k)H^*(k) \\ \quad G(k) = P(k)H^*(k)D^{-1}(k) \\ \quad K(k+1, k) = \phi(k+1, k)G(k) \\ \quad \tau(k) = G(k)H(k) \\ \quad \bar{\tau}(k) = I - \tau(k) \\ \quad \psi(k+1, k) = \phi(k+1, k)\bar{\tau}(k) \\ \text{end loop} \end{array} \right.$$

Define now the block-diagonal matrices $P = \text{diag}[P(1), \dots, P(N)]$, $D = \text{diag}[D(1), \dots, D(N)]$, $G = \text{diag}[G(1), \dots, G(N)]$, $\tau = \text{diag}[\tau(1), \dots, \tau(N)]$, and $\bar{\tau} = \text{diag}[\bar{\tau}(1), \dots, \bar{\tau}(N)]$. Note that then

$$D = HPH, \quad G = PH^*D^{-1}, \quad \tau = GH, \quad \bar{\tau} = I - \tau$$

Also define

$$K = \mathcal{E}_\phi G, \quad \text{and} \quad \mathcal{E}_\psi \triangleq \mathcal{E}_\phi \bar{\tau} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \psi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \psi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \psi(n, n-1) & 0 \end{pmatrix}$$

The transition matrix ψ is defined as

$$\psi = (I - \mathcal{E}_\psi)^{-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \psi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi(N,1) & \psi(N,2) & \dots & I \end{pmatrix}$$

where

$$\psi(k, m) = \prod_{i=m}^{k-1} \psi(i+1, i)$$

These matrices will be used in the sequel to establish the following identities. **Lemma 3:** The matrices S and P are related by

$$S = P + \tilde{\phi}P + P\tilde{\phi}^* + \phi KDK^*\phi^* \quad (8.3)$$

Proof: Observe that

$$\bar{\tau}P\bar{\tau}^* = \bar{\tau}P$$

Then

$$P = \mathcal{E}_\psi P \mathcal{E}_\psi^* + M = \mathcal{E}_\phi P \mathcal{E}_\phi^* + M - KDK$$

Pre and post multiplying with ϕ and ϕ^* leads to the result. ■

Lemma 4: The system inertia matrix \mathcal{M} factors as $[I + \mathcal{K}]D[I + \mathcal{K}]^*$, with the causal matrix \mathcal{K} and the diagonal matrix D defined as

$$\mathcal{K} = H\tilde{\phi}G = H\phi K$$

Proof: Multiply Eq. (8.3) by H and H^* and recall that $D = HPH^*$. ■

Lemma 5: The “open-loop” and “closed-loop” transition matrices ϕ and ψ are related by

$$\psi^{-1} = \phi^{-1} + KH$$

Proof: Observe that

$$\psi^{-1} = I - \mathcal{E}_\psi = I - \mathcal{E}_\phi \bar{\tau} = \phi^{-1} + \mathcal{E}_\phi \tau = \phi^{-1} + KH \quad \blacksquare$$

Lemma 6: The lower triangular factor $I + \mathcal{K}$ can be inverted as

$$(I + \mathcal{K})^{-1} = I - \mathcal{L}$$

in which \mathcal{L} is the lower triangular matrix

$$\mathcal{L} = H\psi K$$

This also implies that $\mathcal{K} = \mathcal{L} + \mathcal{K}\mathcal{L}$, $\mathcal{K} = \mathcal{L} + \mathcal{L}\mathcal{K}$, and $\mathcal{L}\mathcal{K} = \mathcal{K}\mathcal{L}$.

Proof: Observe from Lemma 5 that $\phi - \psi = \psi KH\phi$, and so

$$(I - \mathcal{L})(I + \mathcal{K}) = I - H\psi K + (I - H\psi K)H\phi K = I \quad \blacksquare$$

The above sequence of results are the necessary ingredients to establish the recursive factorization of the inverse of the composite system inertia matrix as in Eq. (8.2).

9 Filtering and Smoothing

Typically, the composite system inertia matrix is inverted to solve what is referred to as the forward dynamics problem. This problem consists of computing a set of joint angle accelerations given a corresponding set of applied joint moments. The joint-angle accelerations $\ddot{\theta}$ and the applied joint moments T are related by

$$\ddot{\theta} = (I - \mathcal{L}^*)D^{-1}(I - \mathcal{L})T$$

where $\ddot{\theta} = [\ddot{\theta}(1), \dots, \ddot{\theta}(N)]$ is the vector of joint-angle accelerations. The factorization of the inverse of the inertia matrix can be implemented using recursive filtering and smoothing algorithms. These algorithms determine the joint-angle accelerations from the joint moments by means of a two-stage computation. The first stage represents filtering and is characterized by the factor $(I - \mathcal{L})$. The second stage represents smoothing and is characterized by the factor $(I - \mathcal{L}^*)$.

Inward Filtering of Joint Moments to Produce Innovations

This stage produces an “innovations” process defined as

$$\epsilon = (I - \mathcal{L})T$$

It produces also the filtered state estimate

$$z = \psi KT$$

The components $z(k)$ of $z = [z(1), \dots, z(N)]$ satisfy the Kalman filter equations [1]

$$\left\{ \begin{array}{l} z(0) = 0 \\ \text{for } k = 1 \dots N \\ z(k+1) = \psi(k+1, k)z(k) + K(k+1, k)T(k) \\ \text{end loop} \end{array} \right.$$

The elements $\epsilon(k)$ of the innovations vector $\epsilon = [\epsilon(1), \dots, \epsilon(N)]$ are defined as

$$\epsilon(k) = T(k) - H(k)z(k)$$

Multiplication of the innovations process by the inverse of the diagonal matrix D produces the residuals

$$\nu = D^{-1}\epsilon$$

These residuals are processed in the smoothing stage that follows.

Outward Smoothing of Residuals to Produce Joint-Angle Accelerations

Smoothing corresponds to multiplication of the residuals by the anti-causal factor $(I - \mathcal{L}^*)$ to obtain the joint-angle accelerations, i.e.,

$$\ddot{\theta} = (I - \mathcal{L}^*)\nu \quad (9.1)$$

A spatial difference equation which is based on Eq. (9.1) can be obtained by re-introducing the costate variables defined earlier to be the spatial accelerations $\alpha = [\alpha(1), \dots, \alpha(N)]$ at the N joints. The costate variables α and the residuals ν are related by

$$\alpha = \psi^* H^* \nu \quad (9.2)$$

Use of this in Eq. (9.1) implies that

$$\ddot{\theta} = \nu - K^* \alpha$$

This last relationship expresses the joint angle accelerations in terms of the residuals and the costate variables. Furthermore, Eq. (9.2) can be used to infer [1] that the joint-angle accelerations and the costate variables satisfy the difference equation

$$\left\{ \begin{array}{l} \alpha(N+1) = 0 \\ \text{for } k = N \cdots 1 \\ \quad \ddot{\theta}(k) = \nu(k) - K^*(k+1, k)\alpha(k+1) \\ \quad \alpha(k) = \phi^*(k+1, k)\alpha(k+1) + H^*(k)\ddot{\theta}(k) \\ \text{end loop} \end{array} \right.$$

These equations are referred to as the Bryson-Frazier smoother equations [11]. Their application to problems in robot dynamics is discussed in more detail in [1].

Physical Interpretation of Filtering and Smoothing Algorithms

To examine the physical interpretation, consider the following predictor-corrector form of the filtering and smoothing techniques:

$$z(1) = 0; \quad P(0) = 0 \quad (9.3)$$

for $k = 1 \dots N$

$$P^+(k-1) \rightarrow [I - G(k-1)H(k-1)]P(k-1) \quad (9.4)$$

$$P(k) \rightarrow \phi(k, k-1)P^+(k-1)\phi^*(k, k-1) + M(k) \quad (9.5)$$

$$D(k) \rightarrow H(k)P(k)H^*(k); \quad G(k) \rightarrow P(k)H^*(k)^{-1}(k) \quad (9.6)$$

$$\epsilon(k) \rightarrow T(k) - H(k)z(k) \quad (9.7)$$

$$\nu(k) \rightarrow D^{-1}(k)\epsilon(k) \quad (9.8)$$

$$z^+(k) \rightarrow z(k) + G(k)\epsilon(k) \quad (9.9)$$

$$z(k+1) \rightarrow \phi(k+1, k)z^+(k) \quad (9.10)$$

end loop

The residuals and the Kalman gains are stored in this stage and used as inputs to the smoothing stage.

$$\alpha(N+1) = 0 \quad (9.11)$$

for $k = N \dots 1$

$$\alpha(k) \rightarrow \phi^*(k+1, k)\alpha(k+1) \quad (9.12)$$

$$\ddot{\theta}(k) \rightarrow \nu(k) - G^*(k)\alpha(k) \quad (9.13)$$

$$\alpha(k) \rightarrow \alpha(k) + H^*(k)\ddot{\theta}(k) \quad (9.14)$$

end loop

The filtering process starts at a fictitious joint 0 attached to the tip body. The initial conditions Eq. (9.3) for the filtered state estimate and for the spatial inertia both vanish. The initial state estimate is zero, because there are no external forces and moments acting at the initial joint. Since this knowledge is precise, there is no uncertainty, and the corresponding estimation error covariance is zero. The initial condition for the Riccati equation is therefore

zero. This is from the point of view of estimation theory. From the point of view of mechanics, the spatial inertia at the initial joint is zero because by definition there is no inertia outboard of this joint. From these initial conditions, the filtering equations proceed with the by now classical predictor-corrector architecture of the Kalman filter, as illustrated in Fig. 5.

This correction involves updating the spatial inertia and state estimate. The equation that updates the spatial inertia is Eq. (9.4). The spatial inertia is propagated by means of the prediction step Eq. (9.5) in the Riccati equation. This predicted inertia is used to compute the Kalman gain by means of Eq. (9.6). The prediction step in the Riccati equation reflects the increase in the spatial inertia due to the addition of a body. From the viewpoint of estimation theory, the prediction step reflects increases in the state estimation error covariance, which are due to the build-up of uncertainty that occurs because the inertial forces internal to body k are assumed to be random. Knowledge of the covariance allows for compensation of this uncertainty.

Correction occurs in crossing a joint from one body to the next. The state update equation Eq. (9.9) combines in an optimal sense two estimates for the moment about the joint axis. One of the estimates comes from the previously conducted prediction step. This estimate has an inherent error with a known covariance $P(k)$. The second estimate comes from the measurement $T(k)$ itself and has no error. The error (innovations) term created in Eq. (9.7) represents the difference between the actual moment (measurement) and the predicted moment $H(k)z(k)$. Note that the innovations process has the units of a pure moment. In Eq. (9.9), the innovations process is multiplied by the Kalman gain to form the correction term that is then added to the predicted state estimate to cross joint k . Central to the correction step is the Kalman gain determined from the Riccati equation. The Kalman gain governs the relative weighting between the predicted state estimate $z(k)$ and the correction term $G(k)\epsilon(k)$.

The step of prediction involves crossing body $k + 1$ from joint k to joint $k + 1$. The state prediction equation Eq. (9.10) is used to propagate the state from the outer to the inner joint. The predicted state is used in Eq. (9.7) to compute an innovations process.

An additional outcome of the filtering equations is the residual process, defined by

Eq. (9.8) in terms of the innovations process $\epsilon(k)$ and the scalar $D(k)$ about the joint k axis. The scalar $D(k)$ can be shown to be [1] the “articulated” inertia of [23, 24]. The articulated inertia $D(k)$ about joint axis k is the scalar inertia of the multibody sub-system outboard of joint k assuming that the inward joints are locked and the outboard joints are free. It is therefore the inertia of a very “floppy” fictitious manipulator where the joints outboard of any given joint allow free rotation. The residual at joint k is an estimate of the joint k angular acceleration, under the assumption that all of the future joints are locked. There is an inherent potential error in this estimate, because the assumption that the accelerations of joints in the future may not be valid. However, this error is compensated for in the smoothing stage that follows.

The smoothing stage also fits within a predictor-corrector architecture which begins at the base with the terminal condition Eq. (9.11). This is shown also in Fig. 5. The inputs to the smoothing stage are a set of residuals. The Kalman gains are also assumed to be known. The outputs are a set of joint-angle accelerations. Prediction occurs in crossing a body by means of the costate propagation equation Eq. (9.12). This equation propagates the costate outwards from joint $k + 1$ to joint k . The joint-angle accelerations $\ddot{\theta}(k)$ are given in terms of the predicted costate by Eq. (9.13). Correction occurs in crossing a joint in an outward direction by means of the costate update equation Eq. (9.14). The joint-angle accelerations $\ddot{\theta}(k)$ computed in Eq. (9.13) are the final solution to the forward dynamics problem. The acceleration at any given joint is that due to past, present, and future joint moments. This is in contrast to the joint accelerations (residuals) computed in the filtering stage, in which only past and present joint moments are used in computing the residual acceleration at a given joint.

Observe in Fig. 5 that the filtering and smoothing stages can be viewed as mirror images of each other about a vertical line that cuts the diagram in two. This is a graphical illustration of the result in Eq. (8.2) that the filtering and smoothing algorithms factor the inverse of the composite multibody robot system inertia matrix as $\mathcal{M}^{-1} = (I - \mathcal{L}^*)D^{-1}(I - \mathcal{L})$. This factorization is studied in more detail in [7, 8].

10 Covariance Analysis

The aim here is to develop formulas to compute the covariance of several relevant quantities (state, state estimation error, innovations, etc.) discussed in previous sections. The random field model Eq. (5.4) is assumed as a starting point.

Recall that the composite system inertia matrix \mathcal{M} is the covariance of the measurement process, i. e.,

$$\mathcal{M} = E(TT^*) = HSH^*$$

This result has an interesting interpretation. It states that the collective system behavior, as represented by the system inertia, emerges from the covariance of the output of the random field model Eq. (5.4). It therefore provides a means to compute the inertia matrix numerically by direct simulation of the stochastic model. From such a simulation, the inertia matrix would emerge, without conducting the more traditional manual derivation of the equations of motion. This could be done by generating repeated realizations of the random process W by repeated call to a white-Gaussian random number generator with zero mean value and covariance \mathcal{M} . Use of this random process as an input to the linear system characterized by $H\phi$ would produce the random process T as an output. The mean and covariance of T could then be computed numerically in a statistical sense by repeating the experimental trials a sufficiently large number of times. The statistically determined covariance of T would be the multibody system inertia \mathcal{M} .

Lemma 7: The innovations is a white-noise process with a covariance given by

$$E(\epsilon\epsilon^*) = D$$

Proof: Observe that

$$E(\epsilon\epsilon^*) = (I - \mathcal{L})E(yy^*)(I - \mathcal{L}^*) = (I - \mathcal{L})HSH^*(I - \mathcal{L}^*)$$

Then use $HSH^* = (I + \mathcal{K})D(I + \mathcal{K}^*)$ and $(I + \mathcal{K})(I - \mathcal{L}) = I$. ■

Lemma 8: The covariance $E(e_s, e_s^*)$ of the state estimation error $e_s = X - z$ is given by

$$E(e_s, e_s^*) = \psi M \psi^* = P + \tilde{\psi} P + P \tilde{\psi}^*$$

in terms of the articulated inertia P resulting from the Riccati equation.

Proof: Observe that $e_s = \psi W$ to obtain that

$$E(e_s, e_s^*) = \psi M \psi^*$$

However,

$$M = P - \mathcal{E}_\psi P \mathcal{E}_\psi^*$$

Pre and post multiplying by ψ and ψ^* respectively leads to

$$\psi M \psi^* = \tilde{\psi} P + P \tilde{\psi}^* + P ■$$

Lemma 9: The covariance of the costates is

$$E(\alpha \alpha^*) = \Omega = \Upsilon + \Upsilon \tilde{\psi} + \tilde{\psi}^* \Upsilon, \quad \text{where } \Omega = \psi^* H^* D^{-1} H \psi \quad (10.1)$$

and in which $\Upsilon = \text{diag}[\Upsilon(1), \dots, \Upsilon(N)]$. The diagonal blocks $\Upsilon(k)$ satisfy

$$\left\{ \begin{array}{l} \Upsilon(N+1) = 0 \\ \text{for } k = N \dots 1 \\ \Upsilon(k) = \psi^*(k+1, k) \Upsilon(k+1) \psi(k+1, k) + H^*(k) D^{-1}(k) H(k) \\ \text{end loop} \end{array} \right. \quad (10.2)$$

Furthermore the matrices P and Υ satisfy the identity

$$HP\Upsilon = H$$

Proof: Since $\alpha = \psi^* H^* D^{-1} \epsilon$ and $E(\epsilon \epsilon^*) = D$, then

$$E(\alpha \alpha^*) = \psi^* H^* D^{-1} H \psi = \Omega$$

Moreover, from Eq. (10.2),

$$H^* D^{-1} H = \Upsilon - \mathcal{E}_\psi^* \Upsilon \mathcal{E}_\psi \quad (10.3)$$

Hence, pre and post multiplying by ψ^* and ψ ,

$$\Omega = \Upsilon + \Upsilon \tilde{\psi} + \tilde{\psi}^* \Upsilon$$

Since $HP\bar{\tau}^* = 0$, premultiplying Eq. (10.3) by HP it follows that $HP\Upsilon = H$. ■

The costate covariance matrix Υ has a very interesting physical interpretation [8]. It is the inverse of what is referred to as the operational task space inertia, the composite inertia of the manipulator as seen from its tip. Because of this it plays a significant role [6, 30] in the computations necessary to solve closed-chain problems in which two or more manipulators are moving a commonly grasped object.

Lemma 10: The smoothed state estimation error $e_f = X - f$ and the filtered state estimation error $e_s = X - z$ are related by

$$e_f = (I - P\psi^* H^* D^{-1} H) e_s$$

Their corresponding covariances are related by

$$\mathcal{P} = E(e_f e_f^*) = (I - P\psi^* H^* D^{-1} H) E(e_s e_s^*) (I - P\psi^* H^* D^{-1} H)^* \quad (10.4)$$

Proof: Observe that: $e_f = e_s - P\alpha$; $\alpha = \psi^* H^* D^{-1} \epsilon$; and $\epsilon = H e_s$. ■

Lemma 11: The smoothed error covariance \mathcal{P} can be expressed in terms of the filtered error covariance matrix P emerging from the discrete-step Riccati equation as

$$\mathcal{P} = (P - P\Upsilon P) + (I - P\Upsilon)\tilde{\psi}P + P\tilde{\psi}^*(I - \Upsilon P) \quad (10.5)$$

Furthermore,

$$H\mathcal{P} = 0; \quad \mathcal{P}H^* = 0 \quad (10.6)$$

Proof: Use $He_s H = D$ and $He_s = H\psi P$ in Eq. (10.4) to obtain that

$$\mathcal{P} = E(e_s e_s^*) - P\psi^* H^* D^{-1} H\psi P$$

Eq. (10.6) follows from the identity $HP\Upsilon = H$. ■

Eq. (10.6) states that the joint-axes projection $H\mathcal{P}$ of the smoothed estimation-error covariance \mathcal{P} vanishes. As in Eq. (7.5) this reflects the lack of measurement noise in the measurement Eq. (5.2). This implies that the projection Hf of the estimated state f along the joint axes coincides, in the sense that there is no estimation error covariance, with the projection of the state HX of the state X of the random field model in Eq. (5.1). The filter and smoother therefore provide a set of force and acceleration estimates which correspond exactly to the deterministic joint-angle force and acceleration estimates. There is no residual uncertainty remaining in the joint angle estimates at the end of the filtering and smoothing computations. Uncertainty is however present in the filtered state estimates. This is however natural because of the sequential approach used to process the applied joint moments. The inward filtering process has a built-in uncertainty because the filtered state estimate at any given joint is not based on the “future” joint moments located in that part of the manipulator closer to the base. The mass properties of the future links are also not accounted for in the corresponding Riccati equation. The effect of future applied moments and links is however completely taken into account by the outward smoothing operation [1].

The covariances of many of the relevant quantities in the random-field estimation problem of this paper are given by the following identities:

$$E[WW^*] = M$$

$$E[XX^*] = S = \phi M \phi^* = R + \tilde{\phi} R + R \tilde{\phi}^*$$

$$E[TT^*] = \mathcal{M} = HRH^* + H\tilde{\phi}RH^* + HR\tilde{\phi}^*$$

$$E[zz^*] = \phi KDK^* \phi^* = (R - P) + \tilde{\phi}(R - P) + (R - P)\tilde{\phi}^*$$

$$E[(X - z)(X - z)^*] = \psi M \psi = P + \tilde{\psi} P + P \tilde{\psi}^*$$

$$E[\epsilon\epsilon^*] = D$$

$$E[\nu\nu^*] = D^{-1}$$

$$E[\alpha\alpha^*] = \Omega = \Upsilon + \Upsilon\tilde{\psi} + \tilde{\psi}^*\Upsilon$$

$$E[\tilde{\theta}\tilde{\theta}^*] = \mathcal{M}^{-1}$$

$$= D^{-1} + K^*\Upsilon K + K^*\psi^*(\mathcal{E}_\psi^*\Upsilon K - H^*D^{-1}) + (K^*\Upsilon\mathcal{E}_\psi - D^{-1}H)\psi G$$

$$E[ff^*] = \phi M \phi^* - \psi M \psi^* + P\Omega P$$

$$E[(X - f)(X - f)^*] = \mathcal{P} = \psi M \psi^* - P\Omega P = (P - P\Upsilon P) + (I - P\Upsilon)\tilde{\psi}P + P\tilde{\psi}^*(I - \Upsilon P)$$

$$E[(f - z)(f - z)^*] = P\Omega P = P(\Upsilon + \Upsilon\tilde{\psi} + \tilde{\psi}^*\Upsilon)P$$

$$E[\hat{W}\hat{W}^*] = Q = M - M\Omega M = (M - M\Upsilon M) - M(\Upsilon\tilde{\psi} + \tilde{\psi}^*\Upsilon)M$$

These provide a good summary of the results of this section.

11 Closed-Form Inertia Matrix Inverse

The foregoing results can be used to obtain in closed form a decomposition of the inverse of the composite multibody robot system inertia. This is done in terms of the covariance matrices P and Υ of the previous section.

Lemma 12: The inverse of the system inertia matrix can be expressed as

$$\begin{aligned}\mathcal{M}^{-1} &= D^{-1} + K^*\Upsilon K + K^*\psi^*(\mathcal{E}_\psi^*\Upsilon K - H^*D^{-1}) + (K^*\Upsilon\mathcal{E}_\psi - D^{-1}H)\psi K \\ &= D^{-1} - K^*\Upsilon K + K^*\psi^*(\Upsilon K - H^*D^{-1}) + (K^*\Upsilon - D^{-1}H)\psi K\end{aligned}$$

Alternatively, it can be expressed as $\mathcal{M}^{-1} = D^{-1}HUH^*D^{-1}$ where

$$U = P + P\Upsilon^+P + P\mathcal{E}_\phi^*\psi^*(\bar{\tau}^*\Upsilon^+P - I) + (P\Upsilon^+\bar{\tau} - I)\psi\mathcal{E}_\phi P, \quad \text{where } \Upsilon^+ \triangleq \mathcal{E}_\phi^*\Upsilon\mathcal{E}_\phi$$

Proof: This follows from $\mathcal{M}^{-1} = (I - \mathcal{L}^*)D^{-1}(I - \mathcal{L})$, $\mathcal{L} = H\psi K$ and use of Eq. (9.2). ■

This result is quite similar to that obtained in [1] by more detailed methods. The result has an interesting potential application in robot dynamics analysis and in control design because it leads to equations of motion of the form $\ddot{\theta} = \mathcal{M}^{-1}(\theta)T$. This is potentially a very useful result, since this system of equations is much easier to work with, in simulation and control design, for instance, than the equivalent system $\mathcal{M}\ddot{\theta} = T$. It also leads to easy means to obtain linearized manipulators models using spatial recursions [28]. The above operator equation immediately leads to the following recursive algorithms for evaluation of the inverse \mathcal{M}^{-1} to the multibody system inertia matrix.

Lemma 13: The general element $\mathcal{M}^{-1}(k, i)$ of the inverse to the multibody system inertia matrix \mathcal{M} can be evaluated in the triangular region $1 \leq k \leq i \leq N$ via successive vertical sweeps of diminishing length. A vertical sweep is defined as a sequence generated by varying k from the diagonal in which $k = i$ to the bottom edge of the triangular region in which $k = 1$. The index i is held constant at a fixed value for each vertical sweep. Successive

vertical sweeps are generated by varying i repeatedly from $i = N$ to $i = 1$.

$$\left\{ \begin{array}{l} \Upsilon(N+1) = 0 \\ \text{for } i = N \dots 1 \\ \quad \Upsilon(i) = \psi^*(i+1, i)\Upsilon(i+1)\psi(i+1, i) + H(i)D^{-1}(i)H^*(i) \\ \quad \mathcal{M}^{-1}(i, i) = D^{-1}(i) + K^*(i+1, i)\Upsilon(i+1)K(i+1, i) \\ \quad \left\{ \begin{array}{l} \lambda(i) = K^*(i+1, i)\Upsilon(i+1)\phi(i+1, i) - H(i)\mathcal{M}^{-1}(i, i) \\ \text{for } k = i-1 \dots 1 \\ \quad \mathcal{M}^{-1}(k, i) = \lambda(k+1)K(k+1, k) \\ \quad \lambda(k) = \lambda(k+1)\psi(k+1, k) \\ \text{end loop} \end{array} \right. \\ \text{end loop} \end{array} \right.$$

12 Concluding Remarks

One of the objectives of the paper is to point out the equivalence of recursive dynamics methods and the filtering and smoothing techniques from state estimation theory. In the view of the author, establishing relationships between ideas and concepts that had been previously thought to be unrelated is one of the more interesting efforts that can be made. This typically leads to the discovery of new physical and mathematical insights that would otherwise be difficult to discover.

A closely related objective has been to show that multibody robot dynamics computations can be organized with the very well understood and highly developed framework of filtering and smoothing. Extensive analytical and computational experience exists with such an architecture. There is the reassuring presence of such familiar concepts as Riccati equations, Kalman gains, innovations, etc. The architecture is very easy to understand both mathematically and physically. This is one of the primary reasons for its popularity. In addition, the number of arithmetic operations required to solve the forward dynamics problem by means of the filter and smoother is linear in the number of bodies. Adding a body is as simple as adding a measurement to the Kalman filter. This linear performance compares favorably with more common approaches that first assemble the composite multibody robot system inertia matrix and then invert this matrix. Even though numerical efficiency is not

the central aim of this research, it does turn out that the filtering and smoothing techniques are very efficient for large N . This feature alone provides the motivation for further investigation of the methods.

There must be a fundamental reason for the relationship between random field estimation and robot dynamics. This relationship is not accidental. While this reason has been discussed to some extent in [1], more investigation is required and is under way. This relationship seems to suggest that statistical models may be more appropriate than classical mechanics models to investigate robot behavior. The statistical models, for example, can easily handle configuration constraints and unpredictable transitions between constrained and unconstrained motion [29]. Collisions and time-varying contact between manipulators and objects in the environment are more easily handled also. A more complete investigation of the application of statistical models to the problems of motion planning and control is currently under way.

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LIST OF FIGURES

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NEWTON EULER	LAGRANGE	ESTIMATION THEORY	BOUNDARY VALUE PROBLEM
	Quadratic Langrangian	Random model	
Newton Laws	Compute Variations	Compute Conditional Expectation	Newton Laws
Add Geometric Constraints	Euler Equation	Fredholm Matrix Equation	Two-Point Boundary-Value Problem
Eliminate Constraint Forces	Application to Multibody System	Gohberg-Krein Factorization	Filtering/ Smoothing
$\mathcal{M}\ddot{\theta} = T$	$\mathcal{M}\ddot{\theta} = T$	$\ddot{\theta} = \mathcal{M}^{-1}T$	$\ddot{\theta} = \mathcal{M}^{-1}T$

Fig. 1. Four approaches to multibody robot dynamics

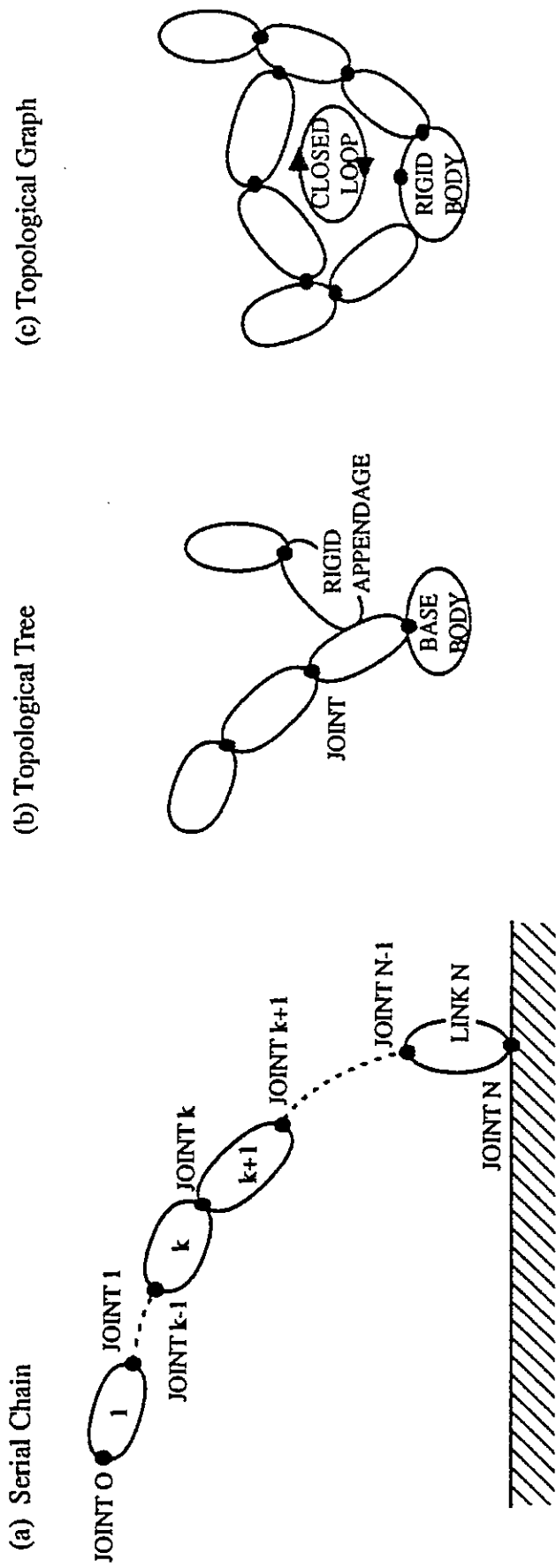


Fig. 2. Typical multibody robotic systems

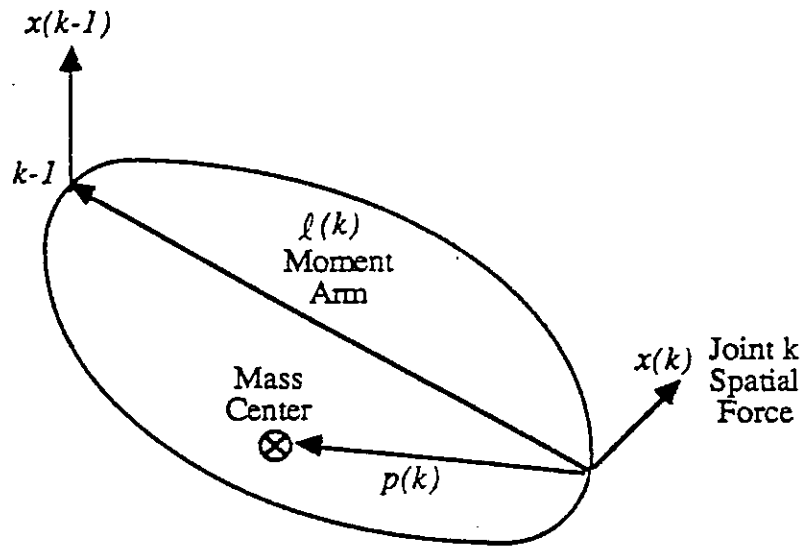


Fig. 3. State space model for typical body

ESTIMATION

States

$X(k)$

Costates

$\alpha(k)$

Measurements

$T(k)$

Transition Matrix

$\phi(k, k-1)$

Process Error Covariance

$M(k)$

Known Deterministic Input

$b(k)$

State-to-Output Map

$H(k)$

DYNAMICS

Spatial Forces

Spatial Accelerations

Active Moments

Intrabody Jacobian

Spatial Inertia

Bias Spatial Force

Projection From State
To Joint Axis

Figure 4. Map between boundary-value problems in estimation and in dynamics

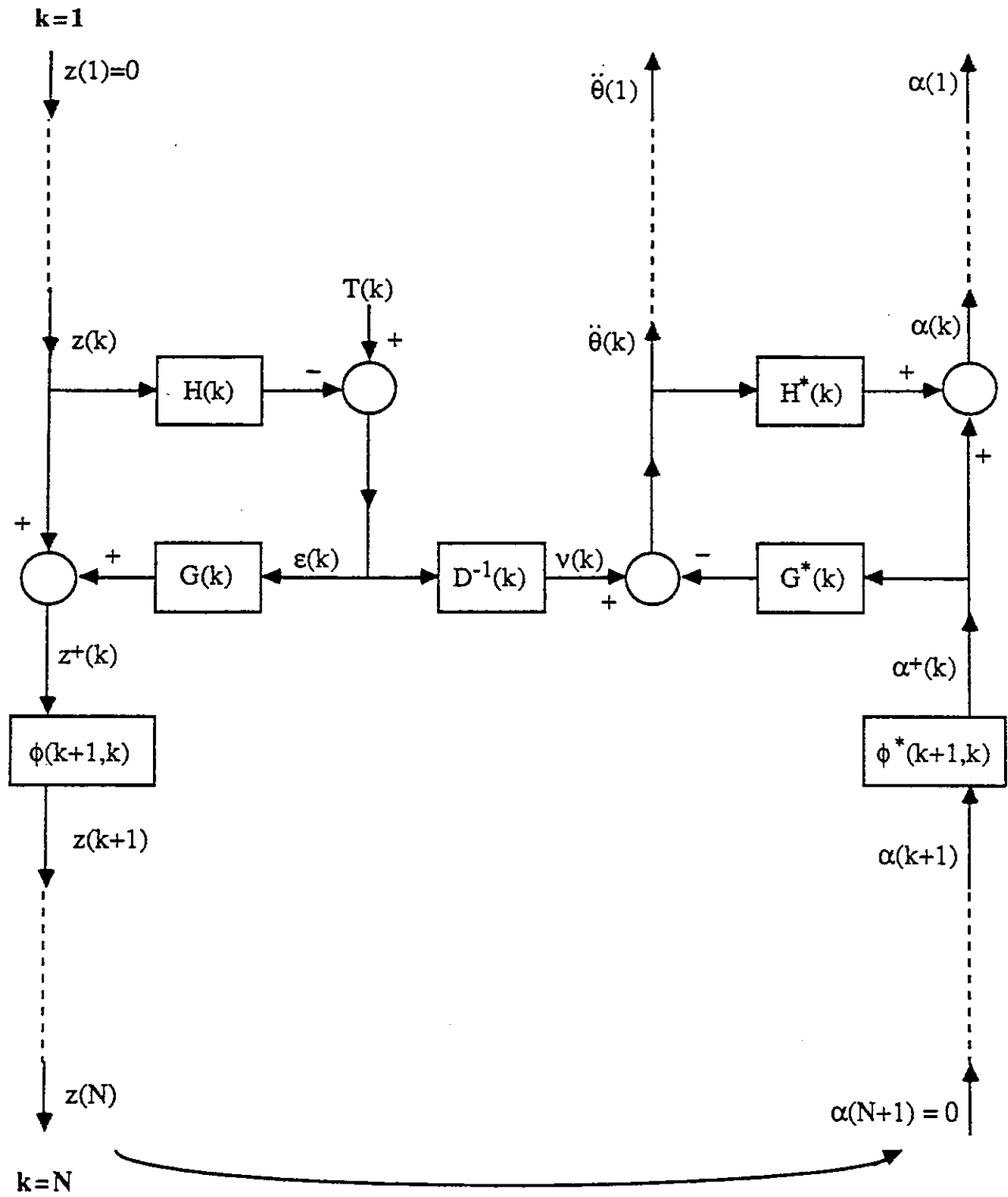


Fig. 5. Filtering and smoothing architecture